ON THE LARGENESS OF THE SUMSETS $A+B$

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MAIN DEFINITION

- $\mathbb{N} := \{0, 1, 2, \dots\}$

- $A \subseteq \mathbb{N}$ is IRREDUCIBLE if

$$A \neq \underbrace{B+C}_{B+C := \{x+y \,:\, x \in B,\, y \in C\}} \quad \text{for all } B, C \subseteq \mathbb{N} \text{ with } |B|, |C| \geq 2.$$

- $A \subseteq \mathbb{N}$ is TOTALLY IRREDUCIBLE if

$$\underbrace{A \cap [n_0, \infty) \neq (B+C) \cap [n_0, \infty)}_{\text{(that is, } A \neq B+C \text{ MODULO FINITE SETS)}} \quad \begin{array}{l} \text{for all } n_0 \in \mathbb{N} \text{ and} \\ \text{for all } B, C \subseteq \mathbb{N} \text{ with } |B|, |C| \geq 2. \end{array}$$

- [IN]FAMOUS OPEN PROBLEM: IS $\mathbb{P} = \{2, 3, 5, \dots\}$ TOTALLY IRREDUCIBLE?
  (OSTMANN'S CONJECTURE)

A KNOWN RESULT

- INFORMAL QUESTION : "$\underbrace{\text{HOW MANY}}$ [TOTALLY] IRREDUCIBLE SETS ARE THERE?"

  $\rightarrow$ QUANTIFIED BY ... ?

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- AN ANSWER: (MEASURE VIEWPOINT) IDENTIFY EACH INFINITE $S = \{s_1, s_2, \dots\} \subseteq \mathbb{N}$ with

$$x = \sum_{n=1}^{\infty} \frac{s_n}{2^n} \in (0,1],$$

- THIS CREATES A BIJECTION FROM $(0,1]$ TO $\underbrace{\{S \subseteq \mathbb{N} : |S| \text{ infinite}\}}_{:= \mathrm{Fin}^+}$

- HENCE WE CAN TRANSFER THE LEBESGUE MEASURE λ FROM $(0,1]$ TO Fin^+.

• QUANTIFIED BY ... ?

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- HENCE WE CAN TRANSFER THE LEBESGUE MEASURE λ FROM $(0,1]$ TO Fin^+.

- THM (WIRSING, 1953): $\lambda\left(\left\{\begin{matrix}\text{TOTALLY} \\ \text{IRREDUCIBLE SETS}\end{matrix}\right\}\right) = 1.$

ANOTHER INTERPRETATION

- INFORMALLY: "DO WE HAVE THE SAME ANSWER TOPOLOGICALLY?"
- CONSIDER $\mathcal{P}(\mathbb{N})$ AS A TOPOLOGICAL SPACE ENDOWED THE PRODUCT TOPOLOGY OF THE DISCRETE TOPOLOGY ON $\{0,1\}$

(HENCE, A BASIC OPEN SET $\mathcal{U} \subseteq \mathcal{P}(\mathbb{N})$ IS OF THE TYPE

$$\mathcal{U} = \{ S \subseteq \mathbb{N} : S \cap [0, n_0] = F \}$$

FOR SOME $n_0 \in \mathbb{N}$ AND SOME $F \subseteq \mathbb{N}$.)

(EQUIVALENTLY $\mathcal{P}(\mathbb{N})$ IS IDENTIFIED WITH THE CANTOR SPACE $\{0,1\}^{\mathbb{N}}$.)

- THEN $\mathcal{P}(\mathbb{N})$ IS A POLISH SPACE

SEPARABLE AND COMPLETELY METRIZABLE

- THEREFORE THE BAIRE CATEGORY THEOREM APPLIES ON $\mathcal{P}(\mathbb{N})$.

$\Downarrow$

$\mathcal{P}(\mathbb{N})$ IS NOT MEAGER IN ITSELF.

"TOPOLOGICALLY SMALL"

($A = \mathbb{Q}$ MEAGER IN $\mathbb{R}$)

[A SET $\mathcal{F} \subseteq \mathcal{P}(\mathbb{N})$ IS MEAGER IF $\mathcal{F} \subseteq \bigcup_{n=1}^{\infty} S_n$, FOR SOME $(S_n)_{n \geq 1}$ CLOSED WITH EMPTY INTERIOR.]

A POSITIVE ANSWER

- THM (L., 2023): $\left\{ \begin{array}{l} \text{TOTALLY} \\ \text{IRREDUCIBLE SETS} \end{array} \right\}^c$ IS MEAGER.

  (THIS IS THE TOPOLOGICAL ANALOGUE OF WIRSING'S THEOREM)

- INFORMALLY: CAN WE SAY SOMETHING "BETTER"?

- FIX $\alpha \in [0,1]$. THEN $A \subseteq \mathbb{N}$ is α-IRREDUCIBLE IF

$$\forall A' = A \text{ modulo } o(x^\alpha) \text{ integers} \implies A' \text{ IRREDUCIBLE.}$$

- ERDÖS CONJECTURE: is $Q = \{1^2, 2^2, 3^2, \dots\}$ $\frac{1}{2}$-IRREDUCIBLE?

  EQUIVALENTLY: IF $A = Q$ modulo $o(\sqrt{x})$ integers, IS A NOT a sumset?

  SÁRKÖZY & SZEMERÉDI PARTIAL ANSWER (1965): Q is $\left(\frac{1}{2} - \varepsilon\right)$-IRREDUCIBLE.

- (SÁRKÖZY, 1962): $\forall A \in \mathrm{Fin}^+$, $\exists A'$ totally irreducible,

$$|(A \,\Delta\, A')(x)| \ll \frac{|A(x)|}{\sqrt{\log\log |A(x)|}}.$$

  COROLLARY: $\forall A \subseteq \mathbb{N}$, $\exists A' = A$ modulo $o(x)$ integers, A' totally irreducible.

A STRONGER POSITIVE ANSWER

- THM (L., 2023): $\forall \alpha < \frac{1}{3}$, $\left\{ \alpha\text{-IRREDUCIBLE SETS} \right\}^c$ is MEAGER.

IDEA OF PROOF: FIX $\alpha \in (0, \frac{1}{3})$.

① TOPOLOGICAL BANACH-MAZUR GAME:

Players I e II choose alternatively nonempty open subsets of $\mathcal{P}(\mathbb{N})$:

$$U_0 \supseteq V_0 \supseteq U_1 \supseteq V_1 \supseteq \ldots\ldots$$

Player II is winner if $\bigcap_{n \geq 1} V_n$ contains an α-irreducible set.

Then $\left\{ \alpha\text{-IRREDUCIBLE SETS} \right\}^c$ MEAGER $\iff$ Player II has a winning strategy.

② EQUIVALENTLY, WE NEED TO DEFINE RECURSIVELY A WINNING STRATEGY OF Player II.

③ RECALL THAT OPEN SETS ARE MADE OF INTERIOR POINTS. HENCE EACH U_i CONTAINS A BASIC OPEN SET

$$\hat{U}_i = \{ S \subseteq \mathbb{N} : S \cap [0, n_i] = F_i \}$$

FOR SOME $n_i \in \mathbb{N}$ AND SOME $F_i \subseteq \mathbb{N}$.

④ FIX $\beta \in \left(\frac{3}{4}, 1\right)$ WITH $\alpha \cdot \beta < \frac{1}{4}$.

⑤ GIVEN $U_0 \supseteq V_0 \supseteq \dots \supseteq U_{i-1} \supseteq V_i \supseteq U_i$, DEFINE

$$V_i = \left\{ S \subseteq \mathbb{N} : S(7n_i + n_i^{2\beta}) = \underbrace{F_i \cup (n_i, 2n_i]}_{\text{"ALMOST A BLOCK OF } 2n_i\text{"}} \cup \underbrace{\bigcup_{j=1}^{n_i^{\beta}} \{5n_i + j n_i^{\beta}\}}_{\text{"A LOT OF HOLES"}} \right\}$$

$$V_i = \left\{ S \subseteq \mathbb{N} : S\left(7n_i + n_i^{2\beta}\right) = \underbrace{F_i \cup (n_i, 2n_i]}_{\text{"ALMOST A BLOCK OF } 2n_i\text{"}} \cup \underbrace{\bigcup_{j=1}^{n_i^{\beta}} \{5n_i + j n_i^{\beta}\}}_{\text{"A LOT OF HOLES"}} \right\}$$

⑥ THIS IS REALLY A WINNING STRATEGY!

DEFINE $\displaystyle\bigcap_{i=1}^{\infty} V_i = \{A\}.$ α - IRREDUCIBLE

FIX $A' = A$ modulo $o(x^{\alpha})$ integers.

SUPPOSE $\boxed{A'} = B + C$ with $B, C \subseteq \mathbb{N}$

$$\Downarrow \; [\ldots !]$$

A' contains at least half of these singletons

$$\Downarrow \; [\ldots !]$$

[CONTRADICTION].

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$\Downarrow$ [. . . !]

A' contains at least half of these singletons

$\Downarrow$ [. . . !]

[CONTRADICTION].

• **COROLLARY:** FIX FAMILIES $\mathcal{A}, \mathcal{B} \subseteq P(\mathbb{N})$ CONTAINING $\{0\}$.

THEN:

$\mathcal{A} + \mathcal{B}$ MEAGER $\Longleftrightarrow$ $\mathcal{A}$ AND $\mathcal{B}$ MEAGER.

ANOTHER STRONGER POSITIVE RESULT

• THM (L., 2023) $\lambda(\{\underbrace{1\text{-IRREDUCIBLE SETS}}_{\substack{\text{After } o(x) \text{ MODIFICATIONS} \\ \text{NEVER SUMSET}}}\}) = 1$.

IDEA OF PROOF:

① $\boxed{\lambda(\Omega) = 1, \quad \Omega := \{\text{NORMAL NUMBERS}\}.}$

② $A = B + C \in \Omega \Longrightarrow B, C$ infinite

③ $A = B + C \in \Omega \Longrightarrow B, C$ have density 0.

④ $\forall n,\ \mathcal{E}_n := \left\{ \underline{X \subseteq [0,n]} : \begin{array}{l} |X \Delta X'|, |Y|, |Z| \leq \frac{n}{17} \\ X' = (Y+Z) \cap [0,n] \\ \text{for some } X', Y, Z \subseteq [0,n] \end{array} \right\}$

$$\begin{cases} ② & A = B + C \in \Omega \implies B, C \text{ infinite} \\ ③ & A = B + C \in \Omega \implies B, C \text{ have density } 0. \end{cases}$$

④ $\forall n,\ \mathcal{E}_n := \left\{ \underline{X \subseteq [0,n]} : \begin{array}{l} |X \triangle X'|, |Y|, |Z| \leq \frac{n}{17} \\ X' = (Y+Z) \cap [0,n] \\ \text{for some } X', Y, Z \subseteq [0,n] \end{array} \right\}$

⑤ $\boxed{\text{ESTIMATE} \quad P_2(\mathcal{E}_n) \ll c^n} \quad (c \in (0,1))$

⑥ $P_2\left(\left\{ \text{NOT 1-IRREDUCIBLE} \right\}\right) \overset{②\&③}{\leq} P_2(\limsup_n \mathcal{E}_n)$

⑤ + Borel-Cantelli.

$= \underline{0}$

OPEN QUESTIONS

1. is $\left\{ \begin{array}{c} 1\text{-IRREDUCIBLE} \\ \text{SETS} \end{array} \right\}^c$ MEAGER?

2. SUPPOSE $A \subseteq \mathbb{N}$ is the set associated with a normal number.

   is it true that $A \neq B+C$ (non trivially)?

3. Is it true that the family of sumsets

$$\{A+B : A, B \subseteq \mathbb{N},\ |A|, |B| \geq 2\}$$

can be regarded as a Borel subset of $[0,1]$?

ASYMPTOTIC [UPPER] DENSITIES ON $\mathbb{N}$

- UPPER ASYMPTOTIC DENSITY of $S \subseteq \mathbb{N}$:

$$d^*(S) := \limsup_{n \to \infty} \frac{|S \cap [0,n)|}{n}.$$

- ASYMPTOTIC DENSITY of $S \subseteq \mathbb{N}$:

$$d(S) := \lim_{n \to \infty} \frac{|S \cap [0,n)|}{n}$$

PROVIDED THAT THE LIMIT EXISTS!

DENOTE BY $\mathrm{dom}(d)$ THE DOMAIN OF d (THAT IS, THE FAMILY OF ALL $S \subseteq \mathbb{N}$ WHICH ADMITS ASYMPTOTIC DENSITY)

THEN

① $\mathrm{dom}(d)$ is NOT an algebra of sets

② d is finitely additive on $\mathrm{dom}(d)$

③ $\mathrm{dom}(d)$ is meager but it has full Lebesgue measure.

SUMSETS VS [UPPER] DENSITIES

INFORMAL PRINCIPLE: "Sumsets are just a few. They have a rather strong structure since

$$A+B = \bigcup_{k \in B} (A + \{k\}).\text{"}$$

"USUALLY" ONE CANNOT USE PROBABILISTIC METHOD TO SHOW EXISTENCE OF SUMSETS WITH SPECIAL PROPERTIES"

INFORMAL (TYPE OF) QUESTIONS: HOW DO THE SUMSETS RELATE WITH d^* OR d? FOR INSTANCE, DO THEY ADMIT ANY PRESCRIBED VALUE OF ASYMPTOTIC DENSITY?

PRELIMINARY QUESTION: HOW TO "QUANTIFY" THE LARGENESS OF $S \subseteq \mathbb{N}$?

$\mathbb{N} := \{0, 1, 2, \dots\}$

EXAMPLES:

(MERTENS)

① THE SET $\mathbb{P}$ of PRIMES HAS LOGARITHMIC DENSITY 0.

(LANDAU)

② THE SET OF INTEGERS $\{x^2 + y^2 : x, y \in \mathbb{N}\}$ HAS ASYMPTOTIC DENSITY 0.

(SZEMEREDI)

③ IF $S \subseteq \mathbb{N}$ HAS POSITIVE UPPER BANACH DENSITY, THEN IT CONTAINS ARBITRARY LONG ARITHMETIC PROGRESSIONS.

EXPLICIT EXAMPLES
(UPPER & LOWER DENSITIES)

$\forall S \subseteq \mathbb{N}$

① UPPER LOGARITHMIC DENSITY

$$ld^*(S) = \limsup_{x \to +\infty} \frac{\sum_{n \in S, n \in [1,x]} \frac{1}{n}}{\log(x)}$$

THE "LOWER" VERSIONS REPLACE limsup WITH liminf

② UPPER ASYMPTOTIC DENSITY

$$d^*(S) = \limsup_{x \to +\infty} \frac{|S \cap [0,x]|}{x}$$

③ UPPER BANACH DENSITY

$$bd^*(S) = \lim_{n \to \infty} \max_{k \geq 0} \frac{|S \cap [k, k+n)|}{n}$$

THE "LOWER" VERSION REPLACES max with MIN

• WHAT ABOUT ~~UPPER~~ ~~LOWER~~ "DENSITIES"?

EXPLICIT EXAMPLES
(DENSITIES)

- **IF** UPPER & LOWER VALUES COINCIDE **THEN** $S \subseteq \mathbb{N}$ ADMITS "DENSITY".

e.g.: ASYMPTOTIC DENSITY $d(S) = \lim_{x \to +\infty} \dfrac{|S \cap [0,x]|}{x}$

THE DOMAIN OF d IS NOT $\mathcal{P}(\mathbb{N})$. INDEED IF

dom (d)

$$S = \bigcup_{n \geq 1} [(2n)!, (2n+1)!]$$

THEN $d^*(S) = 1$ AND $d_*(S) = 0$.

- CONNECTION BETWEEN UPPER & LOWER VERSIONS:

$$\forall S \subseteq \mathbb{N} \qquad d_*(S) = 1 - d^*(\mathbb{N} \setminus S).$$

AN AXIOMATIC DEFINITION

- A FUNCTION $\mu^*: \mathcal{P}(\mathbb{N}) \to \mathbb{R}$ IS AN <u>UPPER DENSITY</u> IF:

INTEGERS

(F1) $\mu^*(\mathbb{N}) = 1$.

(F2) $\mu^*(A) \leq \mu^*(B)$ FOR ALL $A \subseteq B$.

(F3) $\mu^*(A \cup B) \leq \mu^*(A) + \mu^*(B)$ FOR ALL A, B

(F4) $\mu^*(K \cdot A) = \frac{1}{K} \cdot \mu^*(A)$ FOR ALL A AND $K \geq 1$.

(F5) $\mu^*(A + h) = \mu^*(A)$ FOR ALL A AND $h \geq 1$.

$\{Kx : x \in A\}$

- EXAMPLES OF UPPER DENSITIES INCLUDE
  - ALL THE PREVIOUS ONES,
  - THE UPPER POLYA DENSITY,
  - THE UPPER BUCK DENSITY,
  - SOME <u>ADDITIVE</u> FUNCTIONS
  - ETC.

LOWER DENSITIES AND DENSITIES

- LET μ^* BE AN UPPER DENSITY.
- DEFINE ITS ASSOCIATED LOWER DENSITY AS

$$\begin{aligned} \mu_*: \; & P(\mathbb{N}) \longrightarrow \mathbb{R} \\ & S \longmapsto 1-\mu^*(\mathbb{N}\setminus S). \end{aligned}$$

- DEFINE ITS INDUCED DENSITY AS THE RESTRICTION

$$\begin{aligned} \mu: \; & \mathcal{D} \longrightarrow \mathbb{R} \\ & S \longmapsto \mu^*(S), \end{aligned}$$

dom (d) is "NOT" AN ALGEBRA

WHERE $\mathcal{D} = \{ S \subseteq \mathbb{N} : \mu^*(S) = \mu_*(S) \}$

(NOTE THE $\mathcal{D} = \mathrm{dom}(\mu)$ CAN BE AN "UGLY" SUBSET OF $P(\mathbb{N})$)

A "SPECIAL" UPPER DENSITY

- UPPER BUCK DENSITY

$$(\forall S \subseteq \mathbb{N})$$

FINITE UNIONS OF INFINITE ARITHMETIC PROGRESSIONS

$$b^*(S) := \inf\{d(A) : A \in \mathcal{A} \text{ and } S \subseteq A\}$$

ASYMPTOTIC DENSITY.

THEN:

- b^* IS AN UPPER DENSITY.

- IF μ^* IS AN UPPER DENSITY THEN

$$\forall S \subseteq \mathbb{N}, \quad b_*(S) \leq \mu_*(S) \leq \mu^*(S) \leq b^*(S).$$

LOWER BUCK DENSITY

UPPER BUCK DENSITY

BASIC PROPERTIES

SUMSET

$\left(A+B := \{x+y : x \in A, y \in B\}\right)$

Let μ^* be an upper density. Then:

① $\mu(F) = 0 \quad \forall F \subseteq \mathbb{N}$ FINITE.

DENSITY INDUCED BY μ^*

② $\mathcal{A} \subseteq \operatorname{dom}(b) \subseteq \operatorname{dom}(\mu)$.

BUCK DENSITY (INDUCED BY b^*)

FINITE UNIONS OF INFINITE ARITHMETIC PROGRESSIONS

IN ADDITION, $\forall S \in \operatorname{dom}(b) \quad b(S) = \mu(S)$.

③ IF $A \in \mathcal{A}$ THEN $A+B \in \operatorname{dom}(b)$ FOR ALL $B \subseteq \mathbb{N}$.

④ $b(S) = 0 \iff \left(S \text{ COVERS } o(n!) \text{ RESIDUE CLASSES MOD } n!\right)$.

BASIC QUESTION

"WHAT ARE THE POSSIBLE μ-VALUES OF SUMSETS $A+B$, UNIFORMLY IN THE CHOICE OF THE DENSITY μ?"

(so, implictly: $A+B \in \bigcap_{\mu \text{ DENSITY}} \mathrm{dom}(\mu)$.)

RELATED EXAMPLES:

- (VOLKMANN, 1957) $\forall n \geq 2$, $\forall \alpha_1, \dots, \alpha_n, \beta \in [0,1]$ with $\sum \alpha_i \leq \beta$,

  $\exists X_1, \dots, X_n \subseteq \mathbb{N}$, $\big(d(X_1) = \alpha_1, \dots, d(X_n) = \alpha_n$, and $d(X_1 + \dots + X_n) = \beta \big)$.

  ASYMPTOTIC DENSITY

  (exists & =)

ASYMPTOTIC DENSITY

$d(X_1 + \cdots + X_n) = \beta$).

• (NATHANSON, 1990) $\forall n \geq 2$, $\forall \alpha_1, \ldots, \alpha_n, \beta \in [0,1]$ with $\sum \alpha_i \leq \beta$,

$\exists\, X_1, \ldots, X_n \subseteq \mathbb{N}$ $\left(d_*(X_1) = \sigma(X_1) = \alpha_1, \ldots, d_*(X_n) = \sigma(X_n) = \alpha_n, \right.$

and

$\left. d_*(\sum X_i) = \sigma(\sum X_i) = \beta \right).$

LOWER ASYMPTOTIC DENSITY

SCHNIRELMANN DENSITY $\quad \sigma(S) = \inf_{n \geq 1} \frac{|S \cap [1,n]|}{n}$

$\overbrace{X + \cdots + X}^{K \text{ TIMES}}$

$$\begin{cases} d(X) = \frac{\alpha}{n} \\ d(2X) = \frac{2\alpha}{n} \\ \vdots \\ d(nX) = \alpha \end{cases}$$

• (FAISANT, GREKOS, PANDEY, SOMU, 2021) $\forall n \geq 1$ $\forall \alpha \in [0,1]$

$\exists\, X \subseteq \mathbb{N}$, $\quad d(KX) = \frac{K\alpha}{n}$ for all $K = 1, \ldots, n$.

0.91x

Tocca per agganciare a 1x

$d(2X)=\frac{2\alpha}{n}$

$\vdots$

$d(nX)=\alpha$

- (FAISANT, GREKOS, PANDEY, SOMU, 2021) $\forall n \geq 1 \quad \forall \alpha \in [0,1]$

$$\exists X \subseteq \mathbb{N}, \quad d(kX) = \frac{k\alpha}{n} \text{ for all } k=1,\dots,n.$$

- (FAISANT, GREKOS, PANDEY, SOMU, 2021) $\forall \alpha \in [0,1], \ \forall B \subseteq \mathbb{N} \ (0<|B|<\infty),$

$$\exists A \subseteq \mathbb{N} \quad d(A+B) = \alpha.$$

THE PROOFS WERE TAILORED FOR THE ASYMPTOTIC DENSITY d BY USING WEYL'S CRITERION FOR EQUIDISTRIBUTION

THIS DOES <u>NOT</u> EXTEND TO "OTHER DENSITIES"...?

FOR INSTANCE, IS IT TRUE REPLACING d WITH THE BANACH DENSITY?

IT WOULD BE MUCH STRONGER SINCE $d^* \leq bd^*$.

THIS DOES NOT EXTEND TO "OTHER DENSITIES"... ?

FOR INSTANCE, IS IT TRUE REPLACING d WITH THE BANACH DENSITY?

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- (HEGYVÁRI, HENNECART, PACH, 2019)

$$\forall \alpha \in [0,1], \ \exists X \subseteq \mathbb{N} \text{ such that:}$$

$$\Big(0 \in X, \ \gcd(X) = 1, \ \mathrm{d}(X) = \alpha, \text{ and } \underbrace{2X = \mathbb{N}}_{\text{IN PARTICULAR}} \Big)$$

$$\mathrm{d}(2X) = 1$$

FIRST MAIN RESULT

DEFINE $\mathcal{A} := \{$ FINITE UNIONS OF INFINITE ARITHMETIC PROGRESSIONS of $\mathbb{N}\}$

$$\mathcal{A}_\infty := \left\{ \left(\bigcup_n A_n\right) \cup F : A_n \in \mathcal{A} \text{ and } F \text{ finite} \right\}$$

(SUMSET: $A+\cdots+A$ k times)

THM (L., TRINGALI, 2022) $\forall n \geq 1,\ \forall \alpha \in [0,1]$,

$$\exists A \in \mathcal{A}_\infty \text{ such that: } \left(kA \in \operatorname{dom}(\mu) \text{ and } \mu(kA) = \frac{k\alpha}{n} \right)$$

for every $k = 1, \ldots, n$,

for every density μ.

IDEA OF THE PROOF : α RATIONAL

- IT IS SUFFICIENT TO PROVE THE CLAIM FOR $\mu = b$.

- Suppose $\alpha = \dfrac{a}{b}$ with $a, b \in \mathbb{N}$, $b \neq 0$.

  Define

$$A := \{0\} \cup \left(nb \cdot \mathbb{N} + \{1, 2, \ldots, a\} \right) \quad \in \cancel{\mathcal{A}}_{\infty}$$

- Then, for each $K \in \{1, \ldots, n\}$, we have:

$$KA = \{0\} \cup \left(nb \cdot \mathbb{N} + \{1, 2, \ldots, Ka\} \right).$$

- Therefore $\left(KA \in \mathrm{dom}(b) \quad \text{and} \quad b(KA) = \dfrac{Ka}{nb} = \dfrac{K\alpha}{n} \right).$

IDEA OF THE PROOF: α IRRATIONAL

① • A REPRESENTATION OF IRRATIONAL NUMBERS:

FIX $\alpha \in [0,1]$ IRRATIONAL AND $n \in \mathbb{N} \setminus \{0\}$.

THEN $\exists$ SEQUENCES OF POSITIVE INTEGERS $(\beta_i)_{i \geq 1}$ AND $(q_i)_{i \geq 0}$ such that

$$\begin{cases} \bullet \ \alpha = \displaystyle\sum_{i=1}^{\infty} \frac{n!\, \beta_i}{q_1 q_2 \cdots q_i} \\ \bullet \ \gcd(q_i, n q_i \cdots q_{i-1}) = 1 \\ \bullet \ \alpha_i := q_1 \cdots q_i \left(\alpha - \displaystyle\sum_{J=1}^{i} \frac{n!\, \beta_J}{q_1 q_2 \cdots q_J} \right) \text{ BELONGS TO } (0,1) \\ \bullet \ n! \text{ DIVIDES } \lfloor q_i \alpha_{i-1} \rfloor \end{cases} \qquad \forall i \geq 1$$

where $\alpha_0 = \alpha$ and $q_0 = 1$.

② • WEAK ADDITIVITY OF BUCK DENSITY b:

$$\forall A, B \in \mathcal{A} \text{ DISJOINT} \quad \forall X, Y \in \mathrm{dom}(b) \text{ with } X \subseteq A,\ Y \subseteq B,$$

FINITE UNIONS OF INFINITE ARITHMETIC PROGRESSIONS

then:

$$\Big(X \cup Y \in \mathrm{dom}(b) \quad \text{and} \quad b(X \cup Y) = b(X) + b(Y) \Big).$$

NOTE THAT THE ANALOGUE STATEMENT FOR THE ASYMPTOTIC DENSITY d is FALSE!

③ • A "ROUGH" ESTIMATE: IF

$$X := q \cdot \mathbb{N} + \{0, 1, \dots, t-1\}$$

$$\emptyset \neq Y \subseteq q \cdot \mathbb{N} + t$$

$$V := p \cdot \mathbb{N} + s$$

with $\gcd(p, q) = 1$, $t \ll q$

THEN For $K = 1, \dots, n$,

$$\left| b^*\big(K((X \cup Y) \cap V)\big) - \frac{Kt}{pq} \right| \leq \frac{1}{pq}.$$

THE SAME CLAIM HOLDS FOR b_*

(NOTE THAT, IF WE WRITE $b(S)$, WE SHOULD BE SURE THAT S IS THE DOMAIN OF b)

$$V := p \cdot \mathbb{N} + s \qquad\qquad t < q$$

THEN

FOR $K = 1, \dots, n$,
$$\left| b^*\left(K\left((X \cup Y) \cap V\right)\right) - \frac{Kt}{pq} \right| \leq \frac{1}{pq}.$$

THE SAME CLAIM HOLDS FOR b_*

(NOTE THAT, IF WE WRITE $b(S)$, WE SHOULD BE SURE THAT S IS THE DOMAIN OF b)

④ GLUE EVERYTHING TOGETHER: USING THE REPRESENTATION

$$\alpha = \sum_{i=1}^{\infty} \frac{n!\, \beta_i}{q_1 q_2 \cdots q_i}$$

DEFINE TWO SUITABLE SEQUENCES (A_i) AND (B_i) in $\cancel{\mathcal{A}}$ WITH THE PROPERTY THAT (for all $K = 1, \dots, n$)

$$b(A_i) \approx \alpha/n, \quad b(B_i) \approx 0, \quad \text{and} \quad b(K(A_i \cup B_i)) \approx K\, b(A_i \cup B_i) \approx K \frac{\alpha}{n}.$$

⑤ VERIFY THAT THE LIMIT $A := \bigcup_i A_i$ SATISFIES THE CLAIM.

SECOND MAIN RESULT

THM (L., TRINGALI, 2021) $\forall \alpha \in [0,1]$, $\forall B \subseteq \mathbb{N}$ finite nonempty,

$$\exists A \in \mathcal{A}_\infty \quad \big(A+B \in \mathrm{dom}(\mu) \text{ and } \mu(A+B)=\alpha \big)$$

(COUNTABLE UNIONS OF INFINITE ARITHMETIC PROGRESSIONS $\cup$ finite sets)

uniformly in the choice of the density μ.

SKETCH of PROOF: • WLOG $\min B = 0$ AND $\alpha \in (0,1)$. $\gamma := \max(B)$.

SKETCH of PROOF : • WLOG $\min B = 0$ AND $\alpha \in (0,1)$. $y := \max(B)$.

• PICK $h, k \geq 1$ such that $\frac{h}{k} < \alpha < \frac{h+1}{k}$ (h "LARGE").

• FIX $C \in \mathcal{A}_\infty \cap \mathrm{dom}(b)$ $\quad b(C) = k\alpha - h$

COROLLARY of PREVIOUS RESULT

INDEED

$\mathrm{Im}(b) = [0,1]$.

• DEFINE $\quad A := \left(k \cdot \mathbb{N} + \{0, 1, \ldots, h-y-1\}\right) \cup \left(k \cdot C + h - y\right)$.

• THEN $\quad b^*(A+B) = b^*\left(k \cdot \mathbb{N} + \{0, \ldots, h-1\}\right) + b^*\left(k \cdot C + h\right)$

$$= \frac{h + b^*(C)}{k} = \frac{h + b(C)}{k} = \alpha.$$

• ANALOGOUSLY FOR b_*.

THIRD MAIN RESULT

THM (L., TRINGALI, 2021) $\forall \alpha \in [0,1]$ $\exists A \subseteq \mathbb{N}$ such that

$$\left(0 \in A,\ \gcd(A) = 1,\ A \in \mathrm{dom}(\mu),\ \mu(A) = \alpha, \text{ and } 2A = \mathbb{N}\right).$$

UNIFORMLY IN THE DENSITY μ.

SKETCH OF PROOF: • Set $Q := \{x^2 + y^2 : x, y \in \mathbb{N}\}$ $(\Longrightarrow 2Q = \mathbb{N})$

• PICK Y such that $b(Y) = \alpha$. (THIS IS POSSIBLE!)

$\boxed{b(Q) = 0}$

• THEN $A := Q \cup Y$ SATISFIES THE CLAIM. INDEED

$$b(A) = b(Y) = \alpha \quad \text{and} \quad 2A = \mathbb{N}.$$

WHAT HAPPENS TO $\mu(A+B)$ IF B IS INFINITE?

- WE KNOW THAT IF B is nonempty finite then, in particular,

$$\{\mu(A+B) : A \subseteq \mathbb{N}\} = [0,1]$$

FOR EVERY DENSITY μ.

- TRIVIAL EXAMPLE: If $B = \underbrace{2 \cdot \mathbb{N}}_{\{\text{EVEN NATURAL NUMBERS}\}}$ THEN

$$\{d(A+B) : A \subseteq \mathbb{N}\} = \{0, \tfrac{1}{2}, 1\}$$

d: ASYMPTOTIC DENSITY

ANALOGOUSLY: IF $B \in \mathcal{A}$ THEN

$$\{ d(A+B) : A \subseteq \mathbb{N} \}$$

IS A DISCRETE SUBSET OF $\{0,1\}$.

WHAT IF B IS "SUFFICIENTLY SPARSE"?

- (CHU, 2022) IF $B = \{ b_n : n \in \mathbb{N} \}$ SATISFIES $\frac{b_{n+1}}{b_n} \to \infty$ THEN

$$\forall \alpha \in [0,1] \ \exists A \subseteq \mathbb{N} \quad d(A+B) = \alpha.$$

CONJECTURE: GAP IN THE PROOF!

LAST MAIN RESULT

THM (L., TRINGALI 2022) IF $b(B)=0$ THEN

$$\forall \alpha \in [0,1] \quad \exists A \in \mathrm{dom}(b), \quad \mu(A+B)=\alpha.$$

for every density μ

B COVERS $o(n!)$ RESIDUES MODULO $n!$ AS $n \to \infty$

(CHOOSE $B=\mathbb{P}$ AND $\mu=d$:))

COROLLARY : $\forall \alpha \in [0,1] \quad \exists A \subseteq \mathbb{N} \quad d(A+\mathbb{P})=\alpha.$

OPEN QUESTION

- FIX $B \subseteq \mathbb{N}$ WITH

"SUFFICIENTLY SPARSE SET" WHICH MAY COVER ALL RESIDUES MODULO K (FOR EVERY K).

$$|B \cap [1,x]| \ll \log(\log(x))$$

IS IT TRUE THAT

$$\forall \alpha \in [0,1], \exists A \subseteq \mathbb{N} \qquad d(A+B) = \alpha \ ?$$

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THANK YOU!